Last time:

Flux integral

Flux integral is a surface integral of a vector field:

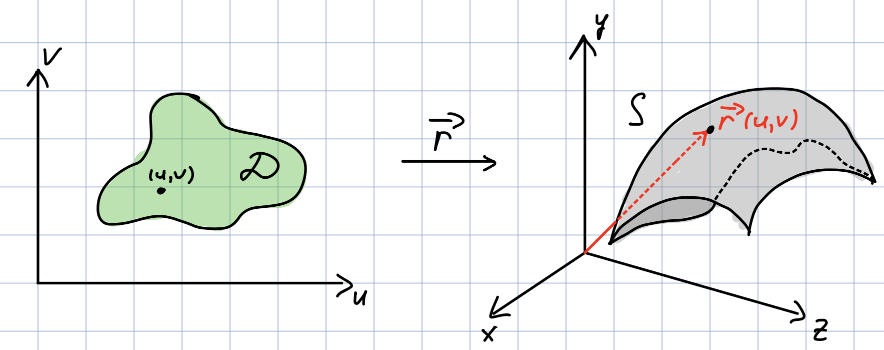
$$\iint_S \vec{F}\cdot d\vec{S} \overset{\text{def}}{=\!=} \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_{\mathcal{D}} \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,\underbrace{dA}_{du\,dv}$$

vector field $\vec{F}(x,y,z)$

unit normal vector field for S

in uv-plane

$$\vec{n} = \frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}$$ - defines an orientation (default orientation induced by parametrization)

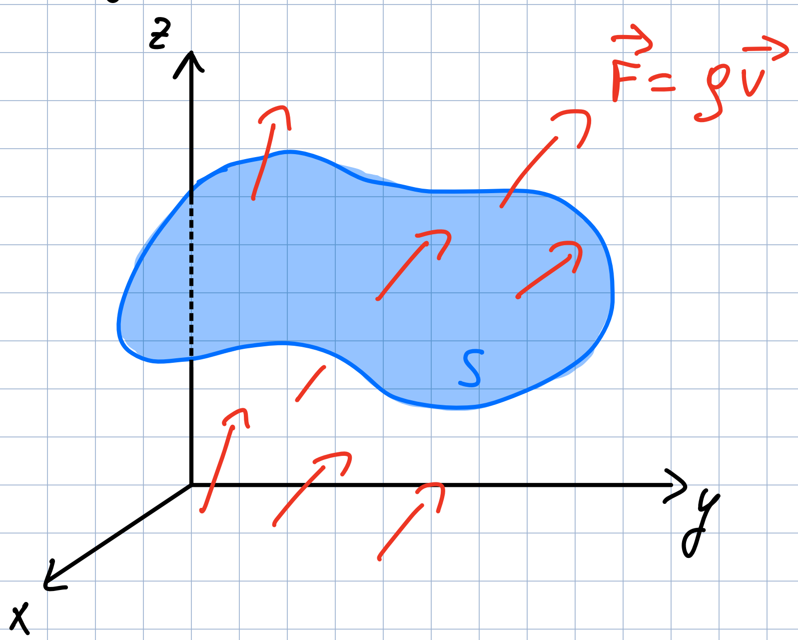


Rmk: If S is a graph of some function $g(x,y)$, then

$$\left.\begin{aligned} &\vec{r}(x,y) = \langle x, y, g(x,y)\rangle \\ &\vec{r}_x\times\vec{r}_y = \langle -g_x, -g_y, 1\rangle \end{aligned}\right\} \Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_{\mathcal{D}} (-Pg_x - Qg_y + R)\, dA$$

assuming upward orientation of S

Physical interpretation:



S - oriented surface with unit normal vector $\vec{n}$

Imagine a fluid with density $\rho(x,y,z)$ and velocity field $\vec{v}(x,y,z)$ flowing through S.

Then the rate of flow (mass per unit time) per unit area is given by $\rho\vec{v}$.

// $\rho\vec{v}\cdot\vec{n}\,A(S_{ij})$ - mass of fluid per unit time crossing S_{ij} in the direction of $\vec{n}$ //

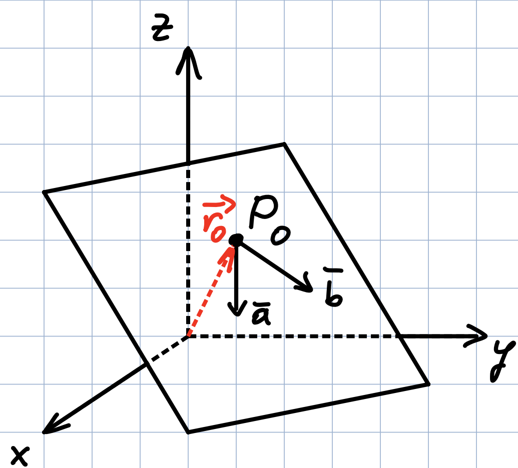
$$\boxed{\iint_S \rho\vec{v}\cdot\vec{n}\,dS \quad \text{- rate of flow through } S.}$$

Find a parametrization $\vec{r}(u,v)$ and an equation of the plane through P_0 containing non-parallel vectors $\vec{a}$ and $\vec{b}$.

$$\boxed{\alpha(x-x_0)+\beta(y-y_0)+\gamma(z-z_0)=0}$$

– scalar equation of the plane through $P_0(x_0,y_0,z_0)$ with normal vector $\vec{n}=\langle\alpha,\beta,\gamma\rangle$

$\vec{n}=\vec{a}\times\vec{b}$



$$\vec{r}(u,v)=\underbrace{\vec{r_0}}_{\langle x_0,y_0,z_0\rangle}+u\underbrace{\vec{a}}_{\langle a_1,a_2,a_3\rangle}+v\underbrace{\vec{b}}_{\langle b_1,b_2,b_3\rangle}$$

or

$$\begin{cases} x = x_0 + ua_1 + vb_1 \\ y = y_0 + ua_2 + vb_2 \\ z = z_0 + ua_3 + vb_3 \end{cases}$$

$P_0 = (2, 2, 3)$

$\vec{a} = \langle 1, 0, 3 \rangle$

$\vec{b} = \langle 2, 3, 0 \rangle$

$P_0 = (1, 3, 4)$

$\vec{a} = \langle 0, 3, 2 \rangle$

$\vec{b} = \langle 1, 0, 2 \rangle$

$P_0 = (7, 3, 9)$

$\vec{a} = \langle 2, 3, 9 \rangle$

$\vec{b} = \langle 0, 1, 0 \rangle$

$P_0 = (6, 4, 7)$

$\vec{a} = \langle 1, 1, 1 \rangle$

$\vec{b} = \langle 0, 2, 3 \rangle$

(a) $P_0 = (2, 2, 3)$ $\vec{a} = \langle 1, 0, 3 \rangle$

$\vec{b} = \langle 2, 3, 0 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{vmatrix} = \langle -9, 6, 3 \rangle$$

$$-9(x-2) + 6(y-2) + 3(z-3) = 0$$

$$\vec{r}(u,v) = \langle 2, 2, 3 \rangle + u\langle 1, 0, 3 \rangle + v\langle 2, 3, 0 \rangle$$

or $\begin{cases} x = 2 + u + 2v \\ y = 2 + 0 + 3v \\ z = 3 + 3u + 0 \end{cases}$

(b) $P_0 = (1, 3, 4)$ $\vec{a} = \langle 0, 3, 2 \rangle$

$\vec{b} = \langle 1, 0, 2 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & 2 \\ 1 & 0 & 2 \end{vmatrix} = \langle 6, 2, -3 \rangle$$

$$6(x-1) + 2(y-3) - 3(z-4) = 0$$

$$\vec{r}(u,v) = \langle 1, 3, 4 \rangle + u\langle 0, 3, 2 \rangle + v\langle 1, 0, 2 \rangle$$

or $\begin{cases} x = 1 + v \\ y = 3 + 3u \\ z = 4 + 2u + 2v \end{cases}$

(c) $P_0 = (7, 3, 9)$ $\vec{a} = \langle 2, 3, 9 \rangle$

$\vec{b} = \langle 0, 1, 0 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 9 \\ 0 & 1 & 0 \end{vmatrix} = \langle 9, 0, 2 \rangle$$

$$9(x-7) + 2(z-9) = 0$$

$$\vec{r}(u,v) = \langle 7, 3, 9 \rangle + u\langle 2, 3, 9 \rangle + v\langle 0, 1, 0 \rangle$$

or $\begin{cases} x = 7 + 2u \\ y = 3 + 3u + v \\ z = 9 + 9u \end{cases}$

(d) $P_0 = (6, 4, 7)$ $\vec{a} = \langle 1, 1, 1 \rangle$

$\vec{b} = \langle 0, 2, 3 \rangle$

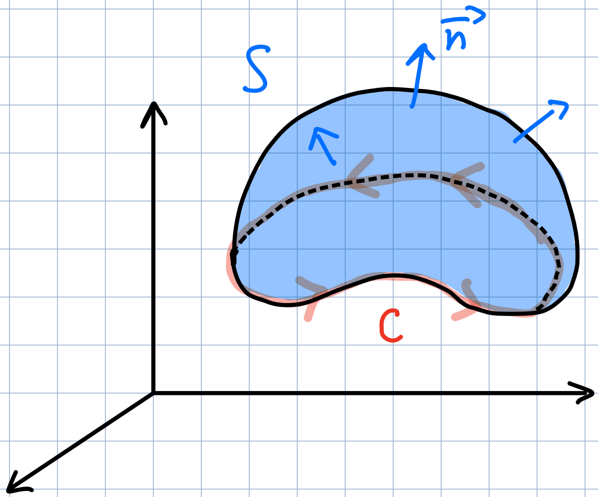
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix} = \langle 3-2, -3, 2 \rangle$$

$$(x-6) - 3(y-4) + 2(z-7) = 0$$

$$\vec{r}(u,v) = \langle 6, 4, 7 \rangle + u\langle 1, 1, 1 \rangle + v\langle 0, 2, 3 \rangle$$

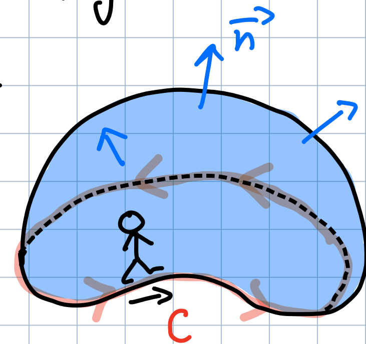
or $\begin{cases} x = 6 + u \\ y = 4 + u + 2v \\ z = 7 + u + 3v \end{cases}$

Stoke's theorem



S - oriented surface (choice of $\vec{n}$),
bounded by a closed curve C.

C has an "induced orientation":
- walking along C with your head in the direction of $\vec{n}$, you should see S on your left.



Stoke's THM:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot \vec{dS}$$

- line integral of (tangental component) of $\vec{F}$ along the boundary equals the flux of curl through the surface.

Notation: $C = \partial S$

Curl: For $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ - vector field,

$$\operatorname{curl}\vec{F} = \underset{\text{mnemonic}}{``\nabla \times \vec{F}"} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

If $\vec{F}$ is conservative, then $\operatorname{curl}\vec{F} = 0$.

Moreover, for $\vec{F}$ on entire $\mathbb{R}^3$, if $\operatorname{curl}\vec{F} = 0$ then $\vec{F}$ is conservative.

$\operatorname{curl}(\nabla f) = \vec{0}$ for any function f.

Divergence:

$$\operatorname{div}\underset{\langle P,Q,R\rangle}{\vec{F}} = \underset{\text{mnemonic}}{``\nabla \cdot \vec{F}"} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$\operatorname{div}(\operatorname{curl}\vec{F}) = 0$ for any vector-field $\vec{F}$.

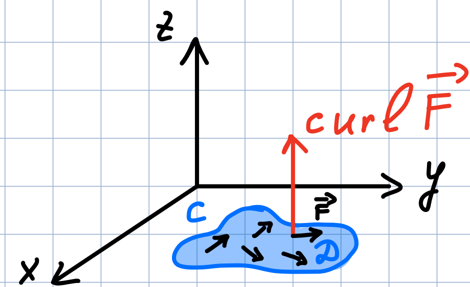
For $\vec{F} = \langle P(x,y), Q(x,y), 0 \rangle$ one has

$$\operatorname{curl}\vec{F} = \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\Rightarrow \vec{k} \cdot \operatorname{curl}\vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$\boxed{\Rightarrow}$ **Green's theorem:**

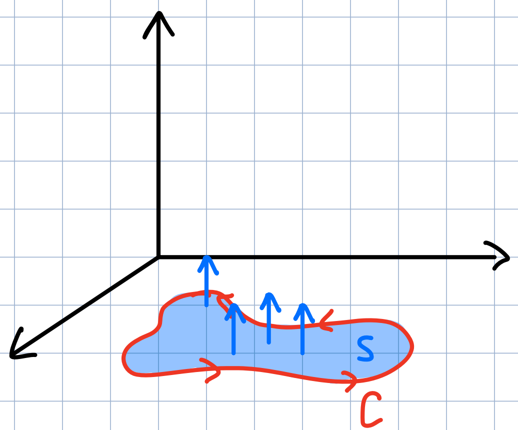
$$\oint_C \vec{F} \cdot d\vec{r} = \int_{\mathcal{D}} \underbrace{(\operatorname{curl}\vec{F}) \cdot \vec{k}}_{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}\, dA$$



$$\operatorname{div}\nabla f = ``\nabla \cdot \nabla f" = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

– Laplace operator.

Special case:



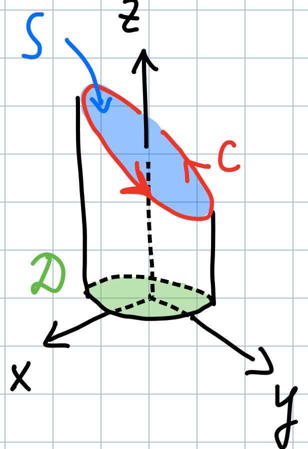
S is flat and lies in xy-plane, $\vec{n} = \vec{k}$, we get

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \underbrace{\operatorname{curl} \vec{F} \cdot \vec{k}}_{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}} \, dS$$

Rmk: We recovered Green's theorem!

Ex: C - intersection of plane $y+z=2$ and cylinder $x^2+y^2=1$.

Find $\oint_C \vec{F}\cdot d\vec{r}$ for $\vec{F}(x,y,z)=\langle -y^2,\ x,\ z^2\rangle$.



Sol: Use Stoke's Thm!

1) $$\text{curl}\,\vec{F}=\vec{\nabla}\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -y^2 & x & z^2\end{vmatrix}=\langle 0,\ 0,\ 1+2y\rangle$$

2) $$S:\ \vec{r}(x,y)=\langle x,\ y,\ 2-y\rangle \qquad (g(x,y)=2-y)$$

$$x,y\in\mathcal{D}\qquad (x^2+y^2\le 1)$$

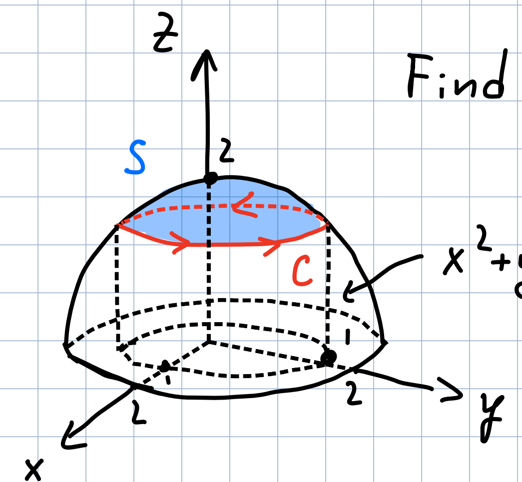
$$\vec{r}_x\times\vec{r}_y=\langle -g_x,\ -g_y,\ 1\rangle=\langle 0,\ 1,\ 1\rangle$$

3) $$\oint_C \vec{F}\cdot d\vec{r}\ \underset{\text{thm}}{\overset{\text{Stoke's}}{=\!=}}\ \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}=\iint_{\mathcal{D}}\langle 0,0,1+2y\rangle\cdot\langle 0,1,1\rangle\,dA=\iint_{\mathcal{D}}(1+2y)\,dA$$

$$\overset{\text{polar}}{=\!=}\int_0^{2\pi}\underbrace{\int_0^1(1+2r\sin\theta)\,r\,dr}_{\frac{1}{2}r^2+\frac{2}{3}r^3\sin\theta\Big|_{r=0}^{r=1}=\frac{1}{2}+\frac{2}{3}\sin\theta}d\theta=\int_0^{2\pi}\left(\frac{1}{2}+\frac{2}{3}\sin\theta\right)d\theta=\frac{1}{2}\theta+\frac{2}{3}\cos\theta\Big|_0^{2\pi}=\pi$$

Ex: S = part of the sphere $x^2+y^2+z^2=4$ inside the cylinder $x^2+y^2=1$, above xy-plane.

Find $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$, where $\vec{F}(x,y,z)=\langle xz, yz, xy\rangle$.



Sol: 1) The boundary of S is the curve C:

$$\begin{cases} x^2+y^2+z^2=4 \\ x^2+y^2=1 \end{cases} \Rightarrow \begin{cases} z=\sqrt{3} \\ x^2+y^2=1 \end{cases}$$ - circle

2) Parametrize C: $\vec{r}(t)=\langle \cos t, \sin t, \sqrt{3}\rangle$

$0 \le t \le 2\pi$

3) $$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} \overset{\text{Stoke's}}{\underset{\text{thm}}{=\!=}} \oint_C \vec{F}\cdot d\vec{r}$$

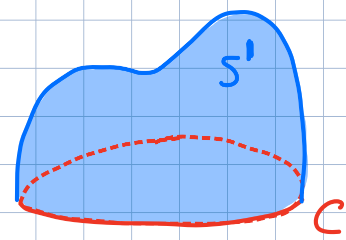
$$= \int_0^{2\pi} \underbrace{\langle \sqrt{3}\cos t, \sqrt{3}\sin t, \sin t\cos t\rangle}_{\langle xz,\, yz,\, xy\rangle} \cdot \underbrace{\langle -\sin t, \cos t, 0\rangle}_{\vec{r}\,'(t)}\, dt =$$

$$= \int_0^{2\pi} \left(\sqrt{3}\cos t(-\sin t) + \sqrt{3}\sin t\cos t + 0\right) dt = 0$$

Rmk: The answer depends only on values of $\vec{F}$ on the boundary curve C!

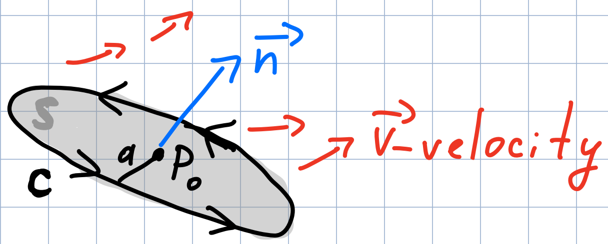
We could have taken another surface S' with same boundary C and

$$\iint_{S'} \operatorname{curl} \vec{F} \cdot d\vec{S} \quad \text{would be the same!}$$



• Interpretation of curl $\vec{v}$

$\uparrow$ velocity field of a fluid



tiny disc about P_0 of radius a.

Stoke's thm

$$\underbrace{\iint_S \operatorname{curl}\vec{v}\cdot\vec{n}\,dS}_{\approx\ \pi a^2 \operatorname{curl}\vec{v}(P_0)\cdot\vec{n}} = \underbrace{\oint_C \vec{v}\cdot d\vec{r}}_{\text{"circulation" of } \vec{v} \text{ around } C}$$

– tendency of fluid to move around C

$$\Rightarrow \operatorname{curl}\vec{v}(P_0)\cdot\vec{n} = \lim_{a\to 0}\frac{1}{\pi a^2}\oint_{C_a}\vec{v}\cdot d\vec{r}$$

The curl of a vector field measures the tendency of that field to rotate around a given point.